Robust Mechanism Design on Networks with Externality Goods

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Abstract

This paper investigates mechanism design for allocating a good with a positive externality without monetary payments, in a setting where each agent knows not only their own valuation but also the valuations of other agents to whom they are connected in a network. The planner's goal is to allocate the good to the agent with the highest valuation. Given the externality — which increases with the valuation of the agent receiving the good — each agent prefers that the good be allocated to the agent with the highest valuation to maximize their own utility. The paper explores the concept of belief-free implementation as proposed by Bergemann and Morris (2009) and demonstrates that the planner can use this partial incentive alignment to accurately identify the agent with the highest valuation. This is achieved by soliciting information from each agent about their own valuation and the valuations of their neighbors, regardless of the agents' beliefs. The paper identifies the network structures that allow for the belief-free implementation of efficient allocation. It proposes a mechanism that ensures efficient allocation, irrespective of agents' beliefs, when the network includes at least one central agent who is connected to all other agents.

Keywords: network, full implementation, belief-free implementation, interdependent valuations, mechanism design without transfers
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1 Introduction

Many allocation problems involve the distribution of goods that benefit not only the recipients but also the entire community. For example, consider a CEO deciding whom to assign as the manager of a department within the company. Among the department members, the CEO wants to give the role to the most productive individual, but she is uncertain who that is. If a highly productive person is chosen for the role, not only does that individual benefit, but the entire department also gains, as a productive leader improves overall performance. In this context, department members may have an incentive to truthfully report the productivity of their colleagues when asked. Another example is a government allocating a budget to subsidize research and development (R&D) for firms within a specific industry. While the subsidy directly benefits the receiving firm by fostering innovation, it also indirectly benefits the entire industry through the spillover effect of new technologies developed by the subsidized firms. If the government wants to identify the most productive firm, it might ask each firm to provide information about their partners, as they may possess relevant insights and they have a partial incentive to tell the truth due to the spillover.

This paper examines the allocation problem faced by a planner, such as a government or CEO, in an environment where the characteristics of each agent (e.g., productivity, ability) are unknown to the planner. Additionally, the good being allocated creates a positive externality, which varies depending on the characteristics of the agents receiving it. This implies that each agent is concerned about who receives the good, even if they do not receive it themselves. I assume the externality factor is represented by $\alpha \in [0, 1]$,meaning that every agent benefits from a fraction α of the output generated by the agents who receive the good. The externality factor, which is independent of who receives the externality, represents a situation where innovations are published through patents, and all firms can access them by paying a cost. Similarly, in an assignment setting, the benefit of a new manager is greater than the benefit each individual department member receives. The planner's objective, in seeking to maximize the sum of utility of agents, hinges on identifying the characteristics of each agent to efficiently allocate the good. This entails optimizing the distribution of the good to ensure that the overall utility among agents is maximized. To achieve this goal, the planner design a mechanism that incentivizes agents to reveal their true information. This strategy helps the planner to identify the characteristics of each agent. The central questions revolve around identifying the conditions under which a mechanism exists that enables the planner to allocate the good efficiently, as well as determining the specific mechanism that achieves this objective.

The answers to these questions depend heavily on the beliefs each agent holds about the information they lack. Since each department member does not necessarily knows the characteristics of all the other members, yet still cares about who receives the good, their best response is shaped by their beliefs about what they do not know. For instance, if an agent believes that there may be a highly productive member that they are unaware of, they might choose to underreport the productivity of a known colleague to increase the likelihood that the more productive, unknown agent will be assigned the role. The typical approach to modeling this situation assumes a common belief that each agent's valuation is drawn from a shared distribution, and it is common knowledge that everyone knows this. While this approach significantly simplifies the model, it is quite restrictive and may not accurately reflect real-world scenarios, especially in our case where the valuation represents productivity, which cannot be assumed to follow the same distribution for everyone. On the other hand, a belief-free approach, which is robust to the underlying belief structure, is much more demanding and, therefore, applicable only in limited contexts (see Bergemann and Morris (2009) and Ollár and Penta (2017)). We address the limitations of the belief-free approach by allowing agents to know the valuations of a subset of other agents, modeling this as a network. In the examples discussed earlier, it is reasonable to assume that a department member may be more familiar with the work habits of others if their desks are nearby or if they have collaborated on projects. If the planner is aware that certain agents possess knowledge about the valuations of others, she can leverage this informational structure to achieve her objectives more effectively.

Taking this context into account, this paper adopts a framework where agents are aware of the valuations of a specific subset of agents within the community. This information structure is represented as a network, where each node corresponds to an agent, and an edge between two nodes indicates that one agent knows the valuation of the other. Given this setup, the central questions can be formalized as follows: What class of networks enables the planner to design a mechanism for efficiently allocating the goods? This paper provides a sufficient condition for networks that allow efficient allocation and proposes mechanisms that achieve efficient allocation regardless of the agents' beliefs.

The challenge in constructing a belief-free mechanism lies in addressing the issue of strategic externalities — when an agent's actions affect other agents that they do not know. In such cases, belief-free efficiency is unlikely to be achieved because if an agent's deviation influences the allocation of others to whom they are not connected, their best response may vary depending on their beliefs. Therefore, the mechanism must ensure one of two things: either the strategic externalities of each agent are restricted to the allocation of agents they are directly connected to, or, if this is not the case, the agent must be able to infer the valuations of the agents whose allocations they can influence.

We address this problem by proposing a straightforward 2-stage mechanism applicable to networks where there is at least one central agent connected to everyone. In the first stage, the central agent reports the valuations of their neighbors and their own. In the second stage, the remaining agents report the valuations of the central agent and their own. The mechanism is designed such that, for the central agent, reporting the valuations of their neighbors untruthfully is weakly dominated in the first stage. Consequently, by assuming that agents do not use weakly dominated strategies, those moving in the second stage can infer that any agents reported with valuations lower than their own have valuations indeed lower than theirs. This design helps maintain the integrity of the information and facilitates efficient allocation.

The first notable feature of the mechanism is its use of the belief-free approach, which is robust to any belief structure. This approach is one of the most resilient concepts in mechanism design and implementation theory, particularly in situations with interdependent preferences where an agent's utility depends on others' valuations. It is thus stronger than other well-known solution concepts. For example, belief-free efficiency is a sufficient condition for ex-post implementability (Crémer and McLean (1985), Bergemann and Morris (2008)), meaning that even after agents learn the true information, they have no incentive to change their reports. This approach also aligns with Bayes-Nash implementability (Jackson (1991)), which assumes a common prior for all agents. Moreover, since the belief-free approach requires that all equilibrium allocations be efficient, it achieves full implementation, unlike partial implementation, which only requires that at least one equilibrium be efficient while permitting the existence of other inefficient equilibria.

Another feature of the mechanism is that it does not involve monetary payments, unlike auction settings. In the examples we discussed, the principal may not be able to use monetary payments to elicit truthful information. Furthermore, if the good being allocated is money itself, as in the case of subsidy allocation, monetary transfers are not applicable. Most discussions on belief-free implementation, such as those in Bergemann and Morris (2009) and Ollár and Penta (2017), focus on scenarios where monetary payments are available. In contrast, our paper addresses the problem by exploring the information structure necessary for achieving belief-free efficiency, specifically focusing on the network structure that allows for such efficiency, rather than devising an optimal monetary payment scheme to incentivize truthful reporting.

This paper contributes to several areas of existing literature. Firstly, it advances the field of implementation theory and mechanism design in environments with interdependent valuations, specifically in contexts without monetary transfers. In such environments, agents' preferences are influenced by the valuations of other agents, and an agent's own valuation is typically unknown to them. This situation introduces complexity where some parameters in an agent's utility function are not directly observable. Previous discussions on this setting have primarily focused on allocation with monetary transfers (Crémer and McLean (1985), Jehiel and Moldovanu (2001)) and, more recently, on scenarios without transfers (Bhaskar and Sadler (2019), Goldlücke and Tröger (2020)). Our paper extends this literature by addressing how belief-free efficiency can be achieved in such environments, specifically through network structures that facilitate this outcome without relying on monetary incentives. In this context, a mechanism forms an incomplete information game, making it crucial to clarify which solution concept to apply. The robustness of each mechanism, particularly regarding its resilience to different belief systems, depends on this choice. The conventional approach often involves assuming that agents have a common prior about the types of other agents, leading to a Bayes-Nash game under a common prior. While this approach offers valuable insights and simplifies analysis by assuming a common prior, it has been criticized for its restrictive assumption.

To address this issue, Bergemann and Morris (2005) introduces the concept of robust implementation, which explores the implementation problem under a belief system governing unknown information, such as the types and beliefs of other agents, as well as their higher-order beliefs. Bergemann and Morris (2009) provides necessary and sufficient conditions for the existence of mechanisms that are robust to any belief system under specific utility functions, though these conditions do not align with our setting. Ollár and Penta (2017) addresses a similar problem by offering conditions on belief restrictions and environments where a monetary transfer scheme can incentivize agents to report their true information. Our contribution is to provide a novel approach to achieving belief-free efficiency by leveraging network structures where agents have information about some other agents. We focus on identifying the network conditions that enable belief-free efficiency, thereby offering a new perspective on achieving robust outcomes without relying on monetary transfers.

Secondly, this paper contributes to the recent literature on mechanism design within information networks, where agents know their own type and the types of their neighbors connected within the network. In Bloch and Olckers (2022), the principal's goal is to establish a complete ranking of agents, with agents' utilities increasing according to their assigned rankings. They show that a necessary and sufficient condition for incentive compatibility is that every pair of agents has a common neighbor. However, their setting does not involve interdependent preferences where agents' utilities depend on others' types, and their objective is not full implementation, allowing for the possibility of inefficient equilibria. Additionally, they use ex-post incentive compatibility as their solution concept, whereas our paper focuses on belief-free efficiency. In Baumann (2023), full implementation is examined in a context where goods are allocated without monetary transfers. In this setting, each agent derives positive utility if they receive the good and zero otherwise, with each agent having a suitability level for receiving the good. The planner's goal is to allocate the good to the most suitable candidate. This scenario differs from ours in that preferences are not interdependent. Moreover, their paper attributes full implementation to partially verifiable information, where agents face a limit on how much they can lie — an aspect we do not incorporate in our model.

2 Model

A principal has 1 unit of divisible good to allocate to n agents. Let $N = \{1, \cdots, n\}$ with $n \geq 2$ be the set of all agents in the community. We denote the good allocated to agent i by x_i, and the allocation profile by $\mathbf{x} = (x_1, \cdots, x_n)$. It is required that for all $i \in N$, $x_i \geq 0$, and $\sum_{i \in N} x_i = 1$ if money burning is not possible, and $\sum_{i \in N} x_i \leq 1$ if it is possible.

Each agent obtains the utility from the allocation profile. The utility function of agent i is as follows:

$$u_i(\mathbf{x}; \mathbf{v}) = (1 - \alpha) v_i x_i + \alpha \sum_{j \in N} v_j x_j$$

where $v_i \in [0, 1]$ represents the valuation of agent i drawn from an unknown distribution. $\alpha \in (0, 1]$ represents the externality factor. We assume that $v_i \neq v_j$ for any $i, j \in N$. In the example of the R&D subsidy allocation problem, we can consider x_i as the amount of subsidy allocated to firm i, v_i as the productivity of firm i, and α as the degree of spillover of the innovation. In the task allocation problem within a team, x_i can be considered as the probability that the task will be allocated to agent i, and α represents the portion of the reward awarded to the entire team.

The planner does not know the type of any agent, while an agent knows his own type and the types of some other agents directly connected into a network. A network is represented by a matrix $\mathbf{G} = (g_{ij})_{i,j \in N}$, where $g_{ij} = 1$ implies that agent i knows v_j, and we assume that the network is undirected. I define N_i as the set of neighbors of agent i. The information possessed by agent i can be summarized as $\theta_i(\mathbf{v}, \mathbf{G}) = (v_i, (v_j)_{j \in N_i}) \in \Theta_i(\mathbf{G})$. For sake of simplicity, we omit the notation for the dependence of the information type on the network and write $\theta_i = \theta_i(\mathbf{v}, \mathbf{G})$ when there is no confusion. The network $\mathbf{G}$ and the externality factor α is a common knowledge among agents and the planner. The objective of the planner is to allocate the integrity of the good to the agent whose valuation is the highest. We call this allocation an efficient allocation. The objective of the paper is to determine under what conditions and how the planner can allocate the good efficiently, despite lacking knowledge of the valuation profile by constructing a mechanism.

Mechanism and robust implementation

We consider a mechanism with message space $M_i = \Theta_i(\mathbf{G})$ and allocation rule $\mathbf{x}(\mathbf{m})$. Let $\mathbf{v} = (v_1, \cdots, v_n)$ be the true realization of valuation profile and $\theta_i = (v_i, (v_j)_{j \in N_i})$ be the information that agent i possesses. We consider a general belief system with no common prior. Let $b_i(\cdot|\theta_i) \in B_i(\theta_i) \subseteq \Delta(V_{-i})$ be his conditional belief on others' type, where

$$B_i(\theta_i) = \{b_i(\cdot) \in \Delta(V_{-i}) \,|\, b_i(\{v'_{-i} \in V_{-i} \,|\, v'_j = v_j \text{ for } j \in N_i\}) = 1\}$$

Agent i's expected payoff when he plays m_i given his conjecture $\mu_i \in \Delta(M_{-i} \times V_{-i})$ is

$$E\pi_i(m_i; \theta_i, \mu_i) = \int_{M_{-i} \times V_{-i}} u_i(\mathbf{x}(m_i, m_{-i}); \mathbf{v}) d\mu_i(m_{-i}, v_{-i}|\theta_i)$$

and the best response is

$$BR_i(\theta_i, \mu_i) = argmax_{m'_i \in M_i} \, E\pi_i(m'_i; \theta_i, \mu_i)$$

The concept of robust implementation is based on the iterative elimination of never-best responses. Let $C_i(B_i(\theta_i))$ be the set of conjectures consistent with his conditional belief, i.e. $C_i(B_i(\theta_i)) = \{\mu_i \,|\, marg_{V_{-i}} \, \mu_i \in B_i(\theta_i)\}$. Let $R_i^0 = M_{-i} \times V_{-i}$, and define inductively

$$R_i^k(\theta_i) = \{(m_i, v_i) \,|\, m_i \in BR_i(\theta_i, \mu_i) \text{ for some } \mu_i \in C_i(B_i(\theta_i)) \cap R_{-i}^{k-1}(\theta_i)\}$$

and let $R_i(\theta_i) = \cap_{k \geq 0} R_i^k(\theta_i)$. The set of rationalizable messages is

$$S_i(\theta_i) = \{m_i \,|\, (m_i, v_i) \in R_i(\theta_i)\}$$

Definition 1. *Given the network* $\mathbf{G}$*, the allocation rule* $\mathbf{x}(\cdot)$ *is robustly efficient if for all* $\mathbf{v}$*,* $\mathbf{x}(\mathbf{m})$ *is efficient for all* $\mathbf{m} \in \times_{i \in N} S_i(\theta_i)$*.*

This concept of rationalizability was first proposed by Battigalli and Siniscalchi (2003) and later incorporated into implementation theory by Bergemann and Morris (2009) and Ollár and Penta (2017). It is known that the set of rationalizable message profiles corresponds to those that can be played as a Bayes-Nash equilibrium under some belief system. Thus, robust efficiency implies that any profile that can be realized as a Bayes-Nash equilibrium in some belief system is efficient, meaning the outcome remains efficient regardless of the specific belief structure.

3 Robustly efficient mechanism

In this section, we propose a mechanism that is robustly efficient and explore the network structures to which it can be applied. We begin by examining the simplest network: two agents connected to each other. We will analyze which mechanisms achieve robust efficiency in this basic setting.

Lemma 1. *Assume that $N = \{1,2\}$ and they are connected. The direct mechanism $(M_1, M_2, \mathbf{x})$ such that*

$$\begin{gathered} m_1^1 > m_1^2 \text{ and } m_2^1 > m_2^2 \Rightarrow (x_1, x_2) = (1,0) \\ m_1^1 < m_1^2 \text{ and } m_2^1 < m_2^2 \Rightarrow (x_1, x_2) = (0,1) \\ \text{Otherwise} \Rightarrow (x_1, x_2) = \left(\frac{\alpha}{1+\alpha}, \frac{\alpha}{1+\alpha}\right) \end{gathered}$$

is robustly efficient.

Proof. w.l.o.g, let $v_1 > v_2$. Let m_i be such that $m_i^1 > m_i^2$ and m_i' be such that $m_i^{1\prime} < m_i^{2\prime}$ for $i \in \{1,2\}$. We will check that for agent 1, m_1' is strictly dominated. We have

$$\begin{aligned} &\pi_1(m_1, m_2) = v_1 \\ &\pi_1(m_1', m_2) = \frac{\alpha}{1+\alpha} v_1 + \frac{\alpha^2}{1+\alpha} v_2 < v_1 \text{ if } v_1 > v_2 \end{aligned}$$

Moreover,

$$\begin{aligned} &\pi_1(m_1, m_2') = \frac{\alpha}{1+\alpha} v_1 + \frac{\alpha^2}{1+\alpha} v_2 \\ &\pi_1(m_1', m_2') = \alpha v_2 < \frac{\alpha}{1+\alpha} v_1 + \frac{\alpha^2}{1+\alpha} v_2 \text{ if and only if } v_1 > v_2 \end{aligned}$$

Therefore, m_1' is strictly dominated.
Once m_1' is eliminated, we can check that for agent 2 m_2' is strictly dominated. We have

$$\begin{aligned} &\pi_2(m_1, m_2) = \alpha v_1 \\ &\pi_2(m_1, m_2') = \frac{\alpha}{1+\alpha} v_1 + \frac{\alpha^2}{1+\alpha} v_2 < \alpha v_1 \text{ if and only if } v_1 > v_2 \end{aligned}$$

Hence, m_2' is strictly dominated. □

When there are conflicting reports between two agents who know each other, and only one unit of the good is to be allocated, allocating $\frac{\alpha}{1+\alpha}$ of the good to each agent and burning the remaining $1 - \frac{2\alpha}{1+\alpha}$ units ensures that the agent with the lower valuation will truthfully report their valuation, while the agent with the higher valuation will also remain truthful. The amount of the good that needs to be burned decreases as α increases. Specifically, if $\alpha = 1$, there is no need to burn any of the good, even if there are conflicting reports. This aligns with the idea that α represents how closely aligned the incentives of the principal and the agents are: the larger α is, the more aligned the incentives become. When $\alpha = 1$, the objective functions of the principal and the agents are identical, eliminating any incentive for agents to report untruthfully.

Building on this lemma, I propose a 2-stage mechanism that is robustly efficient for networks where there is a central agent connected to everyone.

Mechanism

Without loss of generality, let agent 1 be the agent who is connected to all other $n-1$ agents.

1st stage: Agent 1 reports the valuations of everyone, i.e. $m_1 = (m_1^i)_{i \in N}$. If $m_1^i = max_{i' \in N} m_1^{i'}$ for some $i \neq 1$, then $x_i = 1$. Otherwise, without loss of generality, let agent k be such that $\#\{i \in N \,|\, m_1^k > m_1^i\} = k-2$, i.e. agent 2 is ranked the lowest in m_1, agent 3 is the second lowest etc., and go to the 2nd stage.

2nd stage: All peripherals report the valuation of himself and agent 1. Let $S = \{i \neq 1\,;\, m_i^i > m_i^1\}$. If $S \neq \emptyset$, then define $S_1 = S$ and let $i^1 = argmax_{i \in S_1} m_1^i$. Moreover, for each $k \in \{2, \cdots, |S|\}$, define iteratively $S_k = S \setminus \{i^1, \cdots, i^{k-1}\}$ and let $i^k = argmax_{i \in S_k} m_1^i$. The allocation is $x_1 = \left(\frac{\alpha}{1+\alpha}\right)$, $x_{i^k} = \left(\frac{\alpha}{1+\alpha}\right)^k$, and $x_i = 0$ for all $i \neq 1$ and $i \notin S$.

Example 1. Assume that $N = 4$. Since only the order in the report matters, let m_i be the ordering of agents that agent i knows, i.e. $m_1 = (1,2,3,4)$ implies that $m_1^1 > m_1^2 > m_1^3 > m_1^4$. For peripherals, $m_i = i$ implies that $m_i^i > m_i^1$, $m_i = 1$ implies that $m_i^1 > m_i^i$. We show the allocation profiles (x_1, x_2, x_3, x_4) for the cases in which agent 1 reports $(1,2,3,4)$ at the first stage.

$$\begin{aligned}
\mathbf{x}((1,2,3,4),2,3,4) &= \left(\left(\frac{\alpha}{1+\alpha}\right)^3, \frac{\alpha}{1+\alpha}, \left(\frac{\alpha}{1+\alpha}\right)^2, \left(\frac{\alpha}{1+\alpha}\right)^3\right) \\
\mathbf{x}((1,2,3,4),2,3,1) &= \left(\left(\frac{\alpha}{1+\alpha}\right)^2, \frac{\alpha}{1+\alpha}, \left(\frac{\alpha}{1+\alpha}\right)^2, 0\right) \\
\mathbf{x}((1,2,3,4),2,1,4) &= \left(\left(\frac{\alpha}{1+\alpha}\right)^2, \frac{\alpha}{1+\alpha}, 0, \left(\frac{\alpha}{1+\alpha}\right)^2\right) \\
\mathbf{x}((1,2,3,4),2,1,1) &= \left(\frac{\alpha}{1+\alpha}, \frac{\alpha}{1+\alpha}, 0, 0\right) \\
\mathbf{x}((1,2,3,4),1,3,4) &= \left(\left(\frac{\alpha}{1+\alpha}\right)^2, 0, \frac{\alpha}{1+\alpha}, \left(\frac{\alpha}{1+\alpha}\right)^2\right) \\
\mathbf{x}((1,2,3,4),1,3,1) &= \left(\frac{\alpha}{1+\alpha}, 0, \frac{\alpha}{1+\alpha}, 0\right) \\
\mathbf{x}((1,2,3,4),1,1,4) &= \left(\frac{\alpha}{1+\alpha}, 0, 0, \frac{\alpha}{1+\alpha}\right) \\
\mathbf{x}((1,2,3,4),1,1,1) &= (1,0,0,0)
\end{aligned}$$

As observed, the unilateral deviation of agent 4 affects only the allocation of himself and the central agent. This is because he is ranked the lowest in the report provided by agent 1. Consequently, the best response of agent 4 is independent of his beliefs about unknown information, as it does not impact the allocations of agents 2 and 3, whom he does not know.

Another important observation is that for agent 3, a unilateral deviation impacts only the allocations of the central agent, himself, and agent 4, who is ranked lower than agent 3 in the central agent's report. The issue arises because agent 3 does not know the valuation of agent 4. Therefore, if agent 3's deviation affects the allocation of agent 4—whose valuation is unknown to him—the best response of agent 3 might vary depending on his beliefs about agent 4's valuation.

However, the mechanism enables agent 3 to infer the valuation of agent 4 from the central agent's reports. Since reports from the central agent that rank peripheral agents untruthfully are weakly dominated, if the central agent avoids using weakly dominated strategies, peripheral agents can be confident that those ranked lower than themselves by the central agent have lower valuations. This assurance helps agent 3 to make informed decisions, as he can rely on the fact that the central agent's truthful reporting ensures consistency in valuations.

Proposition 1. *Assume that agents do not use weakly dominated strategies. Then the mechanism is robustly efficient.*

Proof. We prove the proposition by the following lemmata.

Lemma 2. *Assume that in m_1 there exists a pair (i, j) such that $v_i > v_j$ but $m_1^i < m_1^j$. Then, m_1 is weakly dominated.*

Proof. Assume $v_i > v_j$. Let m_1 be such that $m_1^i < m_1^j$ and there is no l such that $m_1^i < m_1^l < m_1^j$. Moreover, let m_1' be such that $m_1^{i\prime} > m_1^{j\prime}$ and $m_1^{l_1\prime} > m_1^{l_2\prime}$ if and only if $m_1^{l_1} > m_1^{l_2}$ for all $(l_1, l_2) \neq (i, j)$. Assume first that $i, j \in S$ and that the allocation is $x_i(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{k+1}$ and $x_j(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{k}$. By assumption of m_1', we have $x_i(\mathbf{m}') = \left(\frac{\alpha}{1+\alpha}\right)^{k}$ and $x_j(\mathbf{m}') = \left(\frac{\alpha}{1+\alpha}\right)^{k+1}$ and $x_l(\mathbf{m}') = x_l(\mathbf{m})$ for all other l, hence $\pi_1(\mathbf{m}') > \pi_1(\mathbf{m})$ since $v_i > v_j$. Assume next that either $i \notin S$ or $j \in S$. Then, we have $x_l(\mathbf{m}') = x_l(\mathbf{m})$ for all $l \in N$, hence $\pi_1(\mathbf{m}') = \pi_1(\mathbf{m})$. Therefore, $\mathbf{m}$ is weakly dominated.
Assume now that there exists l such that $m_1^i < m_1^l < m_1^j$. If $v_l > v_i > v_j$, then we can apply the same argument to the pair (l, j). If $v_i > v_j > v_l$, then we can apply the same argument to the pair (i, l). If $v_i > v_l > v_j$, then we can apply the same argument to the pairs (i, l) and (l, j). □

Lemma 3. *Assume that agent 1 does not use weakly dominated strategies. Then, for all $i \neq 1$, the only rationalizable strategy in stage 2 is to report truthfully, i.e. $m_i^1 > m_i^i$ if and only if $v_1 > v_i$.*

Proof. Assume that $v_i > v_{i+1}$ for all $i \neq 1$. Since m_1 is not weakly dominated, it implies that m_1 is such that $m_1^i > m_1^{i+1}$ for all $i \neq 1$. We prove the lemma by induction.

We can check that for agent n, the best response is $m_n^1 < m_n^i$ if and only if $v_1 < v_n$, no matter what strategies the others take. Let m_n and m_n' be such that $m_n^1 < m_n^n$ and $m_n^{1\prime} > m_n^{n\prime}$. Let $\mathbf{m} = (m_n, m_{-n})$ and $\mathbf{m}' = (m_n', m_{-n})$. Since n is ranked at last in m_1, the allocation with $\mathbf{m}$ is $x_n(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{|S|}$ and this implies that $x_1(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{|S|}$. Thus, with $\mathbf{m}'$, the allocation is $x_n(\mathbf{m}') = 0$, $x_1(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{|S|-1}$, and $x_i(\mathbf{m}') = x_i(\mathbf{m})$ for all $i \notin \{1, n\}$. By Lemma 1, for any S, $\pi_n(\mathbf{m}) > \pi_n(\mathbf{m}')$ if and only if $v_n > v_1$.

Next, we prove that for any agent $p \in \{2, \cdots, n-1\}$, if any agent q such that $q > p$ reports truthfully, then the best response for agent p is $m_p^1 < m_p^i$ if and only if $v_1 < v_p$, i.e. the truthful report.

First, assume that $v_1 > v_p$. Note that for any $q > p$, we know that $v_p > v_q$ since agent 1 does not use weakly dominated strategies, hence for all $q > p$, $v_1 > v_q$, and therefore $q \notin S$ since $m_q^1 > m_q^q$ by assumption that they are truthful. Let m_p is such that $m_p^1 < m_p^p$ and m_p' is such that $m_p^{1\prime} > m_p^{p\prime}$. Let $\mathbf{m} = (m_p, m_{-p})$ and $\mathbf{m}' = (m_p', m_{-p})$, and let $S = \{i \neq 1\,;\, m_i^i > m_i^1\}$. We have $x_1(\mathbf{m}) = x_p(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{|S|}$ since for all $q > p$, $q \notin S$. Thus, we have $x_1(\mathbf{m}') = \left(\frac{\alpha}{1+\alpha}\right)^{|S|-1}$, $x_p(\mathbf{m}') = 0$, and $x_i(\mathbf{m}') = x_i(\mathbf{m})$ for all $i \notin \{1, p\}$. By Lemma 1, for any $|S|$, $\pi_p(\mathbf{m}) < \pi_p(\mathbf{m}')$ since $v_1 > v_p$ by assumption.

Now, assume that $v_1 < v_p$. By taking the same definition of $\mathbf{m}$ and $\mathbf{m}'$, and assume that

$$x_i(\mathbf{m}) = \begin{cases} 0 & \text{for } i \notin S \\ \left(\frac{\alpha}{1+\alpha}\right)^{k_i} & \text{for } i \in S \\ \left(\frac{\alpha}{1+\alpha}\right)^{|S|} & \text{for } i = 1. \end{cases}$$

Then, by letting $S^+ = \{i \in S\,;\, i > p\}$ and $S^- = \{i \in S\,;\, i < p\}$, we have

$$x_i(\mathbf{m}') = \begin{cases} 0 & \text{for } i \notin S \text{ and } i = p \\ \left(\frac{\alpha}{1+\alpha}\right)^{k_i - 1} & \text{for } i \in S^+ \\ \left(\frac{\alpha}{1+\alpha}\right)^{k_i} & \text{for } i \in S \setminus S^+ \\ \left(\frac{\alpha}{1+\alpha}\right)^{|S|-1} & \text{for } i = 1 \end{cases}$$

Note that $k_i > k_p$ for all $i \in S^+$. The payoff of agent p in both $\mathbf{m}$ and $\mathbf{m}'$ are

$$\pi_p(\mathbf{m}) = \left(\frac{\alpha}{1+\alpha}\right)^{k_p} v_p + \alpha \left[\sum_{i \in S} \left(\frac{\alpha}{1+\alpha}\right)^{k_i} v_i + \left(\frac{\alpha}{1+\alpha}\right)^{|S|} v_1\right]$$
$$\pi_p(\mathbf{m}') = \alpha \left[\sum_{i \in S^+} \left(\frac{\alpha}{1+\alpha}\right)^{k_i - 1} + \sum_{i \in S^-} \left(\frac{\alpha}{1+\alpha}\right)^{k_i} v_i + \left(\frac{\alpha}{1+\alpha}\right)^{|S|-1} v_1\right]$$

By taking the difference, we obtain

$$\begin{aligned}
&\pi_p(\mathbf{m}) - \pi_p(\mathbf{m}') \\
&= \left(\frac{\alpha}{1+\alpha}\right)^{k_p} v_p + \alpha \left[\sum_{i \in S^+} \left(\frac{\alpha}{1+\alpha}\right)^{k_i - 1} \left(\frac{\alpha}{1+\alpha} - 1\right) v_i + \left(\frac{\alpha}{1+\alpha}\right)^{|S|-1} \left(\frac{\alpha}{1+\alpha} - 1\right) v_1\right] \\
&= \left(\frac{\alpha}{1+\alpha}\right)^{k_p} v_p - \left[\sum_{i \in S^+} \left(\frac{\alpha}{1+\alpha}\right)^{k_i} v_i + \left(\frac{\alpha}{1+\alpha}\right)^{|S|} v_1\right] \\
&= \left(\frac{\alpha}{1+\alpha}\right)^{k_p} \left[v_p - \sum_{i \in S^+} \left(\frac{\alpha}{1+\alpha}\right)^{k_i - k_p} v_i - \left(\frac{\alpha}{1+\alpha}\right)^{|S|-k_p} v_1\right] \\
&> \left(\frac{\alpha}{1+\alpha}\right)^{k_p} \left[v_p - \sum_{i \in S^+} \left(\frac{1}{2}\right)^{k_i - k_p} v_i - \left(\frac{1}{2}\right)^{|S|-k_p} v_1\right] \\
&> \left(\frac{\alpha}{1+\alpha}\right)^{k_p} \left[v_p - \sum_{j=1}^{|S|-k_p} \left(\frac{1}{2}\right)^{j} v_p - \left(\frac{1}{2}\right)^{|S|-k_p} v_1\right] \\
&> \left(\frac{\alpha}{1+\alpha}\right)^{k_p} \left[v_p - \sum_{j=1}^{|S|-k_p} \left(\frac{1}{2}\right)^{j} v_p - \left(\frac{1}{2}\right)^{|S|-k_p} v_p\right] \\
&= \left(\frac{\alpha}{1+\alpha}\right)^{k_p} (v_p - v_p) = 0
\end{aligned}$$

The first inequality comes from $\alpha < 1$, the second one comes from $v_p > v_i$ for all $i \in S^+$, and the third one comes from $v_p > v_1$. □

By Lemma 3, if $1 \neq argmax_{i \in N}\, v_i$, there is always some agent i whose rationalizable strategy is $m_i^i > m_i^1$. In this case, by Lemma 1, the payoff of agent 1 is strictly lower than when agent 1 takes the strategy m_1 such that $m_1^{i^*} = argmax_{i' \in N}\, m_1^{i'}$ for $i^* = argmax_{i' \in N}\, v_{i'}$, with which the allocation is $x_{i^*}(\mathbf{m}) = 1$ and $x_i(\mathbf{m}) = 0$ for all $i \neq i^*$.
If $1 = argmax_{i \in N}\, v_i$, since agent 1 does not use weakly dominated strategies, he reports truthfully. Thus, by Lemma 3, everyone reports truthfully under the rationalizable strategy profile $\mathbf{m}$, and therefore $x_1(\mathbf{m}) = 1$ and $x_i(\mathbf{m}) = 0$ for all $i \neq 1$. □

The belief-free approach requires that the allocation rule impose constraints on how allocations change when an agent deviates from a strategy. For example, if a peripheral agent deviates from a strategy profile and this deviation increases the allocation of a neighboring agent while decreasing the allocation of an agent to whom the peripheral agent is not connected, the peripheral agent must be certain that the valuation of the agent whose allocation decreases is lower than that of the agent whose allocation increases. If this condition is not met, it is possible to construct a belief for the peripheral agent that would make the deviation from the current allocation profile preferable.

In the 2-stage mechanism we propose, we address this issue by allowing peripheral agents to know the ordering of valuations of other agents from the central agent's report in the first stage. Because the central agent does not use weakly dominated strategies, peripheral agents can deduce that all agents ranked lower than themselves in the central agent's report must have lower valuations (Lemma Lemma 2). With this information, the mechanism ensures that when a peripheral agent changes their report, only the allocations of agents ranked lower than them by the central agent are affected. Therefore, the peripheral agent can determine whether a deviation is advantageous or not, independent of their beliefs (Lemma 3).

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